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Elastic and sound orthonormal beams and localized fields in linear mediums: II. Superpositions of longitudinal elastic plane waves in an isotropic medium

George N Borzdov

Department of Theoretical Physics, Belarusian State University, Fr. Skaryny avenue 4, 220050 Minsk, Belarus

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Abstract

Superpositions of longitudinal elastic plane harmonic waves in an isotropic medium are treated using the formalism presented in the previous paper. Intensities and phases of the partial plane waves are specified by the spherical harmonics. The presented solutions describe unique families of elastic fields: orthonormal beams and three different types of localized fields (for the sake of brevity, called storms, whirls and tornadoes). For an elastic storm, the time average energy flux is identically zero at all points. The elastic whirls and tornadoes have circular and spiral energy flux lines, respectively. The solutions are illustrated by calculating fields, energy densities and energy fluxes.

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1. Introduction

In free space or a linear medium, any superposition of plane harmonic waves (eigenwaves) provides an exact solution of the corresponding wave equation. Among them, electromagnetic and weak gravitational orthonormal beams and localized fields, defined by a given set of orthonormal scalar functions on a two- or three-dimensional beam manifold, stand out because of their unique properties [1–3]. The families of orthonormal beams can be used as convenient functional bases in analysis of wave propagation and scattering. The time-harmonic localized fields have a very small (about several wavelengths) core region with maximum intensity of field oscillations and a very specific energy transport [1–3]. In the first paper [4] of this series, we extended the approach proposed in [2] to the case of elastic waves in isotropic and anisotropic mediums and sound waves in an ideal liquid.

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On this basis, we consider here superpositions of longitudinal elastic plane harmonic waves (eigenwaves) in an isotropic medium, defined by the spherical harmonics Y_j^s as [4]

$$\boldsymbol{W}_j^s(\boldsymbol{r}, t) = \exp(-\mathrm{i}\omega t) \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\theta_2} \exp[\mathrm{i}\boldsymbol{r} \cdot \boldsymbol{k}(\theta, \varphi)] Y_j^s(\theta, \varphi) \nu(\theta, \varphi) \boldsymbol{W}(\theta, \varphi) \sin\theta \, \mathrm{d}\theta \tag{1}$$

where $\boldsymbol{W} = \boldsymbol{W}(\theta, \varphi)$ is the amplitude function, and $\nu = \nu(\theta, \varphi)$ is the beam state function. For all fields treated, the value $Y_j^s(\theta, \varphi)$ of function Y_j^s at given θ and φ specifies the intensity and the phase of an eigenwave with the unit wave normal

$$\hat{\boldsymbol{k}} = \boldsymbol{k}/k = \boldsymbol{e}_\mathrm{r} = \sin\theta'(\boldsymbol{e}_1 \cos\varphi + \boldsymbol{e}_2 \sin\varphi) + \boldsymbol{e}_3 \cos\theta' \tag{2}$$

where $\boldsymbol{e}_\mathrm{r}$ is the radial basis vector of the spherical coordinate system, $(\boldsymbol{e}_i)$ are the Cartesian basis vectors, $\theta' = \kappa_0\theta$, and parameter κ_0 satisfies the condition $0 < \kappa_0 \leqslant 1$. These fields are formed from plane waves propagating in the solid angle $\Omega = 2\pi(1 - \cos\kappa_0\theta_2)$.

The plan of the paper is as follows. In section 2, we present two basically different types of orthonormal beams. Time-harmonic localized fields are treated in section 3. Concluding remarks are made in section 4.

2. Orthonormal beams

2.1. Orthonormal beams with $\theta_2 = \pi/2$, $\kappa_0 = 1$, and $\Omega = 2\pi$

Let us consider orthonormal beams $\boldsymbol{W}_j^s$ (1) with $\theta_2 = \pi/2$ and $\kappa_0 = 1$, formed from plane waves propagating into a solid angle $\Omega = 2\pi$. A longitudinal elastic eigenwave has the one-dimensional amplitude subspace, and its displacement vector $\boldsymbol{u}$ can be written as $\boldsymbol{u} = (\boldsymbol{u} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}}$. With $\kappa_0 = 1$, we have $\theta' = \theta$, and the amplitude function becomes [4]

$$\boldsymbol{W}(\theta, \varphi) \equiv \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\mathrm{r} \\ \mathrm{i}k(\lambda_\mathrm{L}\boldsymbol{e}_3 + 2\mu_\mathrm{L}\cos\theta\,\boldsymbol{e}_\mathrm{r}) \end{pmatrix} \tag{3}$$

where $\boldsymbol{f} = \sigma\boldsymbol{e}_3$ is the force density, σ is the stress tensor, $k = 2\pi/\lambda = \omega/v_1$ is the wavenumber, $v_1 = \sqrt{(\lambda_\mathrm{L} + 2\mu_\mathrm{L})/\varrho}$ is the phase velocity, λ_L and μ_L are the Lame modules [5], and ϱ is the medium density.

With $\kappa_0 = 1$ and $\boldsymbol{W}(\theta, \varphi)$ (3), the orthonormalizing function [4] $\nu = \nu(\theta, \varphi)$ reduces to a constant. As a consequence, from equations (1) and (3) we find the displacement vector $\boldsymbol{u}$ and the force density $\boldsymbol{f}$ of the beam to be

$$\boldsymbol{u} = \nu_1 \mathrm{e}^{\mathrm{i}(s\psi - \omega t)} \boldsymbol{u}_0 \tag{4}$$

$$\boldsymbol{f} = \mathrm{i}k\nu_1 \mathrm{e}^{\mathrm{i}(s\psi - \omega t)} \{\mu_\mathrm{L}\boldsymbol{e} I_j^{ss-1}[\sin\circ 2] + \mu_\mathrm{L}\boldsymbol{e}^* I_j^{ss+1}[\sin\circ 2] + \boldsymbol{e}_3 I_j^{ss}[\lambda_\mathrm{L} + 2\mu_\mathrm{L}\cos^2]\} \tag{5}$$

where

$$\boldsymbol{u}_0 = \boldsymbol{e} I_j^{ss-1}[\sin] + \boldsymbol{e}^* I_j^{ss+1}[\sin] + \boldsymbol{e}_3 I_j^{ss}[\cos] \tag{6}$$

$$\nu_1 = \frac{1}{\pi}\sqrt{\frac{N_Q}{\varrho v_1^3}} \qquad \boldsymbol{e} = (\boldsymbol{e}_\mathrm{R} + \mathrm{i}\boldsymbol{e}_\mathrm{A})/2 \tag{7}$$

$$\boldsymbol{e}_\mathrm{R} = \boldsymbol{e}_1 \cos\psi + \boldsymbol{e}_2 \sin\psi \qquad \boldsymbol{e}_\mathrm{A} = -\boldsymbol{e}_1 \sin\psi + \boldsymbol{e}_2 \cos\psi \tag{8}$$

$$\boldsymbol{r} = R\boldsymbol{e}_\mathrm{R} + z\boldsymbol{e}_3 \qquad R = r\sin\gamma \qquad z = r\cos\gamma. \tag{9}$$

Here, N_Q is the normalizing constant [4], R, ψ and z are the cylindrical coordinates, r, γ and ψ are the spherical coordinates of the point with radius-vector $\boldsymbol{r}$. Complex scalar function $I_j^{sm}[f] = I_j^{sm}[f](r, \gamma)$ is defined by the spherical harmonic $Y_j^s = Y_j^s(\theta, \varphi)$, an integer m and a scalar function $f = f(\theta)$. Its definition and the properties are presented in [2, 4]. The notations emphasize the fact that $I_j^{sm}[f](r, \gamma)$ is a functional regarding f at fixed r and γ. For any given f, $I_j^{sm}[f](r,\gamma)$ is a function of r and γ. When it cannot cause a misunderstanding, we omit the arguments (r,γ). The real and imaginary parts of $I_j^{sm}[f]$ can be separated as [2,4]

$$I_j^{sm}[f] = i^{|m|}(J_{j0}^{sm}[f] + \mathrm{i}J_{j1}^{sm}[f]). \tag{10}$$

Using the general relations for elastic eigenwaves [4] and the expression $\boldsymbol{W}_j^s$ (3), we can calculate tensors γ and σ for all partial eigenwaves and then replace $\boldsymbol{W}(\theta,\varphi)$ in (1) by the obtained tensor amplitude functions $\gamma(\theta,\varphi)$ and $\sigma(\theta,\varphi)$. This gives, on integration, the deformation and the stress tensors of the beam as

$$\gamma = \mathrm{i}k\nu_1 \mathrm{e}^{\mathrm{i}(s\psi-\omega t)}\gamma_0 \tag{11}$$

$$\sigma = \mathrm{i}k\nu_1 \mathrm{e}^{\mathrm{i}(s\psi-\omega t)}(\lambda_\mathrm{L}\mathbf{1}I_j^{ss}[1] + 2\mu_\mathrm{L}\gamma_0) \tag{12}$$

where

$$\gamma_0 = \rho I_j^{ss-2}[\sin^2] + \rho^* I_j^{ss+2}[\sin^2] + \rho_1 I_j^{ss}[\sin^2] + \rho_2 I_j^{ss-1}[\sin\circ 2] + \rho_2^* I_j^{ss+1}[\sin\circ 2] + \rho_3 I_j^{ss}[\cos^2] \tag{13}$$

$$\rho = \boldsymbol{e}\otimes\boldsymbol{e} \qquad \rho_1 = \boldsymbol{e}\otimes\boldsymbol{e}^* + \boldsymbol{e}^*\otimes\boldsymbol{e} = \tfrac{1}{2}(\mathbf{1}-\rho_3) \tag{14}$$

$$\rho_2 = \tfrac{1}{2}(\boldsymbol{e}\otimes\boldsymbol{e}_3 + \boldsymbol{e}_3\otimes\boldsymbol{e}) \qquad \rho_3 = \boldsymbol{e}_3\otimes\boldsymbol{e}_3 \tag{15}$$

and $\mathbf{1}$ is the unit dyadic.

Substituting $\boldsymbol{u}$ (4), γ (11) and σ (12) into the expressions for the time average kinetic w_K and elastic w_E energy densities, and the time average energy flux density vector $\boldsymbol{S}$ (see [4, 5] for the definitions of these quantities) yields

$$w_\mathrm{K} = w_0\sum_{p=0}^{1}\{\tfrac{1}{2}(J_{jp}^{ss-1}[\sin])^2 + \tfrac{1}{2}(J_{jp}^{ss+1}[\sin])^2 + (J_{jp}^{ss}[\cos])^2\} \tag{16}$$

$$w_\mathrm{E} = \frac{w_0}{\lambda_\mathrm{L}+2\mu_\mathrm{L}}\sum_{p=0}^{1}\Big\{\lambda_\mathrm{L}(J_{jp}^{ss}[1])^2 + \frac{\mu_\mathrm{L}}{2}[(J_{jp}^{ss-2}[\sin^2])^2 + (J_{jp}^{ss+2}[\sin^2])^2 + (J_{jp}^{ss-1}[\sin\circ 2])^2 + (J_{jp}^{ss+1}[\sin\circ 2])^2 + 2(J_{jp}^{ss}[\sin^2])^2 + 4(J_{jp}^{ss}[\cos^2])^2]\Big\} \tag{17}$$

$$\boldsymbol{S} = S_0\boldsymbol{S}' = \frac{2S_0}{\lambda_\mathrm{L}+2\mu_\mathrm{L}}\mathrm{Re}\ (\lambda_\mathrm{L}\boldsymbol{u}_0^* I_j^{ss}[1] + 2\mu_\mathrm{L}\gamma_0\boldsymbol{u}_0^*) \tag{18}$$

where $S_0 = N_Q/\lambda^2$, $w_0 = S_0/v_1$. The normal component S'_N of the normalized energy flux density vector $\boldsymbol{S}'$ is given by

$$S'_\mathrm{N} = \frac{S_3}{S_0} = \frac{1}{\lambda_\mathrm{L}+2\mu_\mathrm{L}}\sum_{p=0}^{1}\{\mu_\mathrm{L}(J_{jp}^{ss-1}[\sin]J_{jp}^{ss-1}[\sin\circ 2] + J_{jp}^{ss+1}[\sin]J_{jp}^{ss+1}[\sin\circ 2]) + 2J_{jp}^{ss}[\cos]J_{jp}^{ss}[\lambda_\mathrm{L} + 2\mu_\mathrm{L}\cos^2]\}. \tag{19}$$

The radial S'_R, the azimuthal S'_A, and the normal S'_N cylindrical components of $\boldsymbol{S}'$ as well as both energy densities w_K and w_E are independent of the azimuthal angle ψ. Figure 1 shows S'_N as a function of $R' = R/\lambda$ at $z = 0$.

For the beams defined by the zonal spherical harmonics ($s = 0$, $j = 0, 1, \ldots$), the above general relations simplify drastically. In particular, equations (6), (13) and (18) become

$$\boldsymbol{u}_0 = \boldsymbol{e}_\mathrm{R} I_j^{01}[\sin] + \boldsymbol{e}_3 I_j^{00}[\cos] \tag{20}$$

$$\gamma_0 = \tfrac{1}{2}(\boldsymbol{e}_\mathrm{R}\otimes\boldsymbol{e}_\mathrm{R} - \boldsymbol{e}_\mathrm{A}\otimes\boldsymbol{e}_\mathrm{A})I_j^{02}[\sin^2] + \tfrac{1}{2}(\boldsymbol{e}_\mathrm{R}\otimes\boldsymbol{e}_3 + \boldsymbol{e}_3\otimes\boldsymbol{e}_\mathrm{R})I_j^{01}[\sin\circ 2] + \tfrac{1}{2}(\mathbf{1}-\rho_3)I_j^{00}[\sin^2] + \rho_3 I_j^{00}[\cos^2] \tag{21}$$

$$\boldsymbol{S} = S_0\left(S'_\mathrm{R}\boldsymbol{e}_\mathrm{R} + S'_\mathrm{N}\boldsymbol{e}_3\right) \tag{22}$$

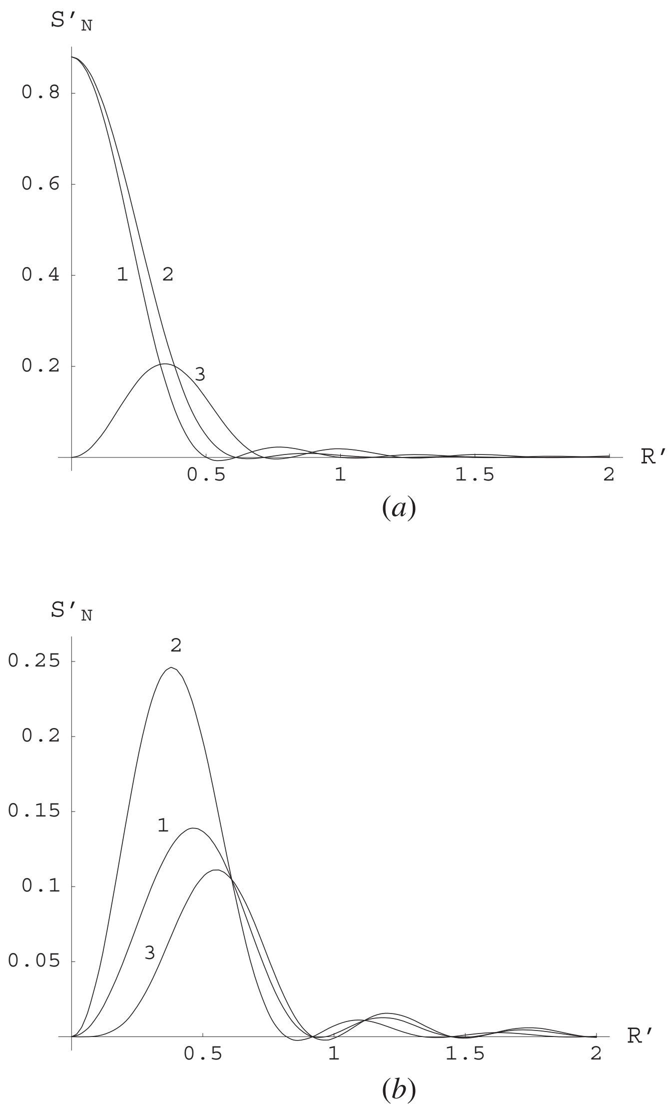

Figure 1. Normal component $S'_{\rm N}$ of the normalized energy flux vector of elastic beams as a function of $R' = R/\lambda$; $z = 0$; $\lambda_{\rm L}/\mu_{\rm L} = 7/9$; $\theta_2 = \pi/2$; $\kappa_0 = 1$; $\Omega = 2\pi$; (*a*) $j = s = 0$ (curve 1); $j = 1$, $s = 0$ (curve 2); $j = s = 1$ (curve 3); (*b*) $j = 2$, $s = 0$ (curve 1); $j = 2$, $s = 1$ (curve 2); $j = s = 2$ (curve 3).

where

$$S'_{\rm R} = \frac{2}{\lambda_{\rm L} + 2\mu_{\rm L}} \sum_{p=0}^{1} \{\mu_{\rm L} J^{00}_{jp}[\cos] J^{01}_{jp}[\sin \circ 2] + J^{01}_{jp}[\sin](J^{00}_{jp}[\lambda_{\rm L} + \mu_{\rm L}\sin^2] + \mu_{\rm L} J^{02}_{jp}[\sin^2])\} \tag{23}$$

$$S'_{\rm N} = \frac{2}{\lambda_{\rm L} + 2\mu_{\rm L}} \sum_{p=0}^{1} \{J^{00}_{jp}[\cos] J^{00}_{jp}[\lambda_{\rm L} + \mu_{\rm L}\cos^2] + \mu_{\rm L} J^{01}_{jp}[\sin] J^{01}_{jp}[\sin \circ 2]\}. \tag{24}$$

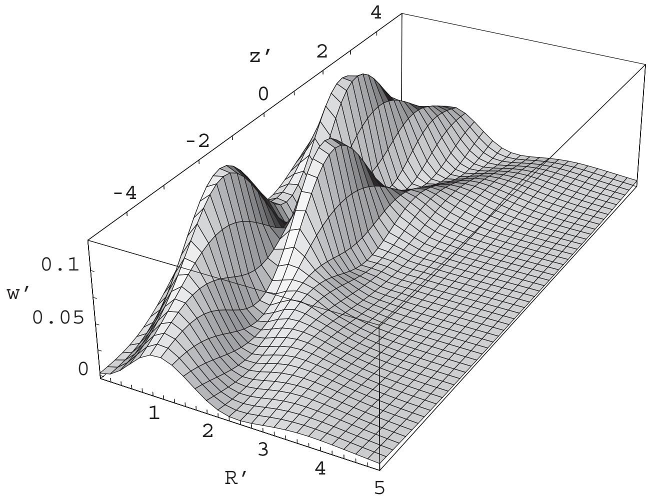

Figure 2. Normalized energy density $w' = (w_{\rm K} + w_{\rm E})/w_0$ of an elastic beam as a function of cylindrical coordinates $R' = R/\lambda$ and $z' = z/\lambda$; $\lambda_{\rm L}/\mu_{\rm L} = 7/9$; $\theta_2 = \pi$; $\kappa_0 = 0.3$; $\Omega = 2\pi[1 - \cos(0.3\pi)]$; $j = 2$; $s = 1$.

Hence, for such beams both the displacement vector $\boldsymbol{u}$ and the energy flux vector $\boldsymbol{S}$ lie in the meridional planes.

2.2. *Orthonormal beams with $\theta_2 = \pi$, $\kappa_0 \leqslant 1/2$, and $\Omega \leqslant 2\pi$*

The spherical harmonics Y_j^s comprise a complete orthonormal system on the unit sphere S^2. However, for the fields treated in the previous section, the beam manifold $\mathcal{B}$ [4] is the northern hemisphere $S^2_{\rm N}$ of S^2. As a consequence these fields form two separate sets of orthonormal beams, defined by the spherical harmonics Y_j^s with even and odd j, respectively. It may be advantageous to obtain a complete system of orthonormal beams $\boldsymbol{W}_j^s$, defined by the whole set of spherical harmonics Y_j^s, for which $\langle \boldsymbol{W}_j^s | Q | \boldsymbol{W}_{j'}^{s'} \rangle = 0$ if $j' \neq j$ and/or $s' \neq s$.

To this end, let us set $\theta_2 = \pi$, $\kappa_0 \leqslant 1/2$, and the amplitude function

$$\boldsymbol{W}(\theta, \varphi) \equiv \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_{\rm r} \\ {\rm i}k(\lambda_{\rm L}\boldsymbol{e}_3 + 2\mu_{\rm L}\cos\theta'\boldsymbol{e}_{\rm r}) \end{pmatrix} \tag{25}$$

with $\boldsymbol{e}_{\rm r}$ given by (2). In this case, the beam manifold is the unit sphere ($\mathcal{B} = S^2$), $\Omega = 2\pi(1 - \cos\kappa_0\pi) \leqslant 2\pi$, and orthonormalizing function [4] becomes

$$\nu(\theta) = \frac{1}{\pi}\sqrt{\frac{\kappa_0 N_Q \sin\kappa_0\theta}{2\varrho v_1^3 \sin\theta}}. \tag{26}$$

The smaller is κ_0, the smaller is Ω, and the beam becomes more collimated. Conversely, if $\kappa_0 = 1/2$, $\Omega = 2\pi$, the beam has a pronounced core region. When $s \neq 0$ and $\kappa_0 = 1/2$, or $\kappa_0 \approx 1/2$, such beams resemble elastic tornadoes with spiral energy fluxes. The energy density distributions for these beams are axially symmetrical with respect to the z axis. Therefore, the transport of energy can be conveniently illustrated by calculated energy densities in a meridional plane. Figure 2 illustrates this energy distribution for the beam defined by the spherical harmonic Y_2^1. It is interesting that the maximum values of the normalized energy density w' near the z axis are reached in the planes $z' = \pm 2$, whereas, in the symmetry plane $z = 0$, w' peaks at a larger ring.

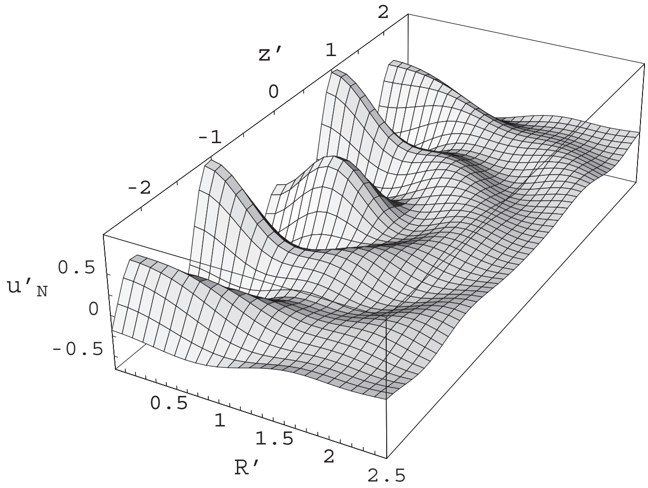

Figure 3. Normal component $u'_{\rm N}$ of the normalized instantaneous displacement field $\boldsymbol{u}' = ({\rm Re}\,\boldsymbol{u})/u_n$ $[u_n = (2/\omega)\sqrt{w_0/\rho}]$ of an elastic storm as a function of $R' = R/\lambda$ and $z' = z/\lambda$; $\lambda_{\rm L}/\mu_{\rm L} = 7/9$; $\theta_2 = \pi$; $\kappa_0 = 1$; $\Omega = 4\pi$; $j = 3$; $s = 0$; $t = 0$.

3. Localized fields

In this section, we consider time-harmonic fields $\boldsymbol{W}^s_j$ (1) with $\pi/2 \leqslant \theta_2 \leqslant \pi$, $\kappa_0 = 1$ $(\theta' = \theta)$ and $2\pi \leqslant \Omega \leqslant 4\pi$. For the sake of simplicity, we assume that the beam state function $\nu = \nu(\theta, \varphi)$ reduces to a constant. Similar electromagnetic and weak gravitational fields were shown to possess many interesting properties [1–3]. Taking into consideration peculiarities of energy transport, one can distinguish three types of such localized fields—termed 'storms' $(\theta_2 = \pi, s = 0)$, 'whirls' $(\theta_2 = \pi, s \neq 0)$ and 'tornadoes' $(\pi/2 \leqslant \theta_2 \leqslant \pi, s \neq 0)$.

3.1. Storms and whirls

If $\theta_2 = \pi$, the fields under consideration are composed of plane waves of all possible propagation directions. They are in effect three-dimensional standing waves with $\mathcal{B} = S^2$ and $\Omega = 4\pi$ (see figures 3 and 4). Substituting the amplitude function $\boldsymbol{W}$ (3) in equation (1) with $\theta_2 = \pi$, we obtain a standing wave with the displacement vector field $\boldsymbol{u}$ and the force density $\boldsymbol{f}$ as

$$\boldsymbol{u} = i^{|s|+q}\sqrt{2}\nu_1 {\rm e}^{{\rm i}(s\psi-\omega t)}\{(-1)^p[\boldsymbol{e}\,\beta(-s)J^{ss-1}_{jp}[\sin] + \boldsymbol{e}^*\beta(s)J^{ss+1}_{jp}[\sin]] + \boldsymbol{e}_3\,J^{ss}_{jq}[\cos]\} \quad (27)$$

$$\boldsymbol{f} = \sigma\boldsymbol{e}_3 = i^{|s|+p+1}\sqrt{2}k\nu_1 {\rm e}^{{\rm i}(s\psi-\omega t)}\{(-1)^q\mu_{\rm L}[\boldsymbol{e}\,\beta(-s)J^{ss-1}_{jq}[\sin\circ 2] + \boldsymbol{e}^*\beta(s)J^{ss+1}_{jq}[\sin\circ 2]] + \boldsymbol{e}_3\,J^{ss}_{jp}[\lambda_{\rm L} + 2\mu_{\rm L}\cos^2]\} \quad (28)$$

where

$$\beta(s) = \begin{cases} -1 & (s = -1, -2, \ldots) \\ 1 & (s = 0, 1, 2, \ldots). \end{cases} \quad (29)$$

Here, $p = 1 - q = 0$ if $j + |s|$ is even, and $p = 1 - q = 1$ if $j + |s|$ is odd.

The deformation γ and the stress σ tensor fields are given by

$$\gamma = i^{|s|+p+1}\sqrt{2}k\nu_1 {\rm e}^{{\rm i}(s\psi-\omega t)}\gamma_0 \quad (30)$$

$$\sigma = i^{|s|+p+1}\sqrt{2}k\nu_1 {\rm e}^{{\rm i}(s\psi-\omega t)}(\lambda_{\rm L}\mathbf{1}J^{ss}_{jp}[1] + 2\mu_{\rm L}\gamma_0) \quad (31)$$

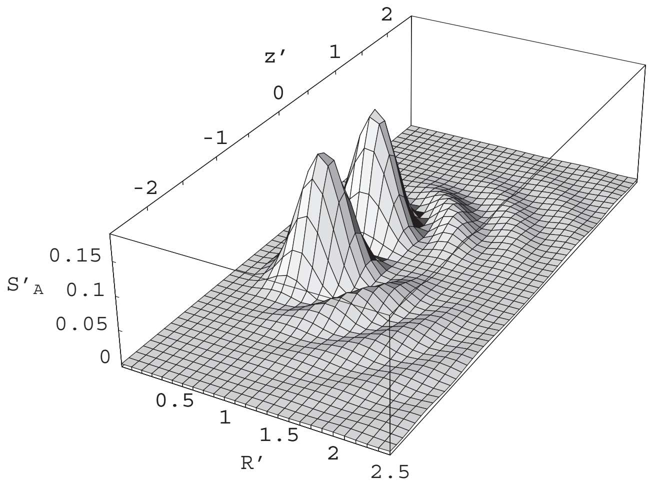

Figure 4. Azimuthal component $S'_{\rm A}$ of the normalized energy flux vector of an elastic whirl as a function of $R' = R/\lambda$ and $z' = z/\lambda$; $\lambda_{\rm L}/\mu_{\rm L} = 7/9$; $\theta_2 = \pi$; $\kappa_0 = 1$; $\Omega = 4\pi$; $j = 4$; $s = 3$.

where

$$\begin{aligned}\gamma_0 = \rho\alpha(s)J_{jp}^{ss-2}[\sin^2] + \rho^*\alpha(-s)J_{jp}^{ss+2}[\sin^2] + \rho_1 J_{jp}^{ss}[\sin^2]\\ +\rho_2(-1)^q\beta(-s)J_{jq}^{ss-1}[\sin\circ 2]\\ +\rho_2^*(-1)^q\beta(s)J_{jq}^{ss+1}[\sin\circ 2] + \rho_3 J_{jp}^{ss}[\cos^2]\end{aligned} \tag{32}$$

with $\alpha(1) = 1$ and $\alpha(s) = -1$ for $s \neq 1$.

Using $\boldsymbol{u}$ (27), γ (30), and σ (31), we obtain the kinetic energy density

$$w_{\rm K} = w_0\{(J_{jp}^{ss-1}[\sin])^2 + (J_{jp}^{ss+1}[\sin])^2 + 2(J_{jq}^{ss}[\cos])^2\} \tag{33}$$

the elastic energy density

$$\begin{aligned}w_{\rm E} = \frac{w_0}{\lambda_{\rm L} + 2\mu_{\rm L}}\{2\lambda_{\rm L}(J_{jp}^{ss}[1])^2 + \mu_{\rm L}[(J_{jp}^{ss-2}[\sin^2])^2 + (J_{jp}^{ss+2}[\sin^2])^2\\ +(J_{jq}^{ss-1}[\sin\circ 2])^2 + (J_{jq}^{ss+1}[\sin\circ 2])^2 + 6(J_{jp}^{ss}[\sin^2])^2]\}\end{aligned} \tag{34}$$

and the energy flux density vector $\boldsymbol{S} = S_0 S'_{\rm A}\boldsymbol{e}_{\rm A}$ which has the only non-vanishing (azimuthal) component given by

$$\begin{aligned}S'_{\rm A} = \frac{2}{\lambda_{\rm L} + 2\mu_{\rm L}}\{\beta(-s)J_{jp}^{ss-1}[\sin][\mu_{\rm L}\alpha(s)J_{jp}^{ss-2}[\sin^2] - J_{jp}^{ss}[\lambda_{\rm L} + \mu_{\rm L}\sin^2]]\\ +\beta(s)J_{jp}^{ss+1}[\sin][J_{jp}^{ss}[\lambda_{\rm L} + \mu_{\rm L}\sin^2] - \mu_{\rm L}\alpha(-s)J_{jp}^{ss+2}[\sin^2]]\\ +\mu_{\rm L}J_{jq}^{ss}[\cos][\beta(s)J_{jq}^{ss+1}[\sin\circ 2] - \beta(-s)J_{jq}^{ss-1}[\sin\circ 2]]\}.\end{aligned} \tag{35}$$

Figure 4 illustrates the azimuthal energy fluxes of the elastic whirl defined by the spherical harmonic Y_4^3. There are two domains with large azimuthal energy fluxes, symmetrical with respect to the plane $z = 0$, whereas in this plane $\boldsymbol{S} \equiv 0$. Calculations show that the whirls, defined by the sectorial spherical harmonics ($s = j$), have only one such domain. For such whirls, the azimuthal energy flux peaks in the plane $z = 0$. Since $S'_{\rm A} = S'_{\rm A}(R', z')$ is independent of the azimuthal angle ψ, all whirls have circular energy flux lines.

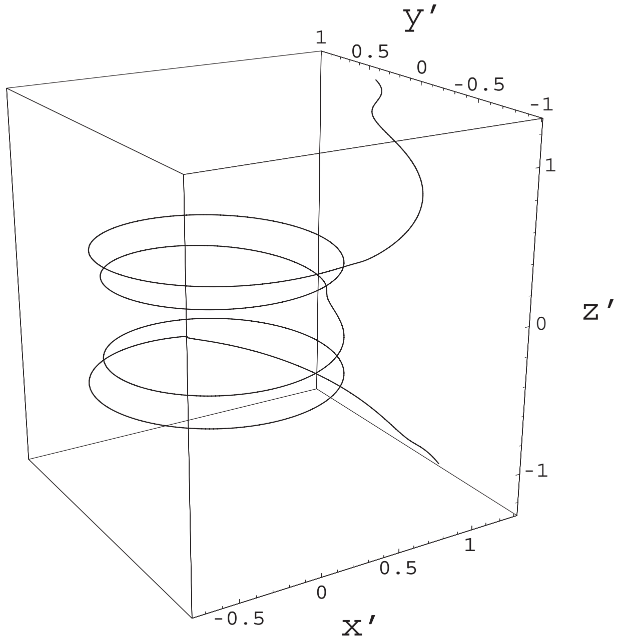

Figure 5. Line of energy flux of an elastic tornado; $x' = x^1/\lambda$; $y' = x^2/\lambda$; $z' = x^3/\lambda$; $\lambda_L/\mu_L = 7/9$; $\theta_2 = 5\pi/6$; $\kappa_0 = 1$; $\Omega = 2\pi(1 + \sqrt{3/2})$; $j = 3$; $s = 2$.

If $s = 0$, equations (27), (28), and (32) reduce to

$$u = i^q\sqrt{2}\nu_1 e^{-i\omega t}\{(-1)^p e_R J^{01}_{jp}[\sin] + e_3 J^{00}_{jq}[\cos]\} \tag{36}$$

$$f = i^{p+1}\sqrt{2}k\nu_1 e^{-i\omega t}\{(-1)^q e_R \mu_L J^{01}_{jq}[\sin\circ 2] + e_3 J^{00}_{jp}[\lambda_L + 2\mu_L\cos^2]\} \tag{37}$$

$$\gamma_0 = \tfrac{1}{2}(e_A \otimes e_A - e_R \otimes e_R)J^{02}_{jp}[\sin^2] + \tfrac{1}{2}(e_R \otimes e_3 + e_3 \otimes e_R)(-1)^q J^{01}_{jq}[\sin\circ 2]$$
$$+\rho_1 J^{00}_{jp}[\sin^2] + \rho_3 J^{00}_{jp}[\cos^2]. \tag{38}$$

In this case, the displacement vectors lie in the meridional planes, and the time average energy flux is everywhere zero. The non-vanishing cylindrical components of the displacement vector u (36), i.e. the normal component u_N and the radial component u_R, are independent of the azimuthal angle ψ. Figure 3 illustrates instantaneous field of the normal component u_N for the case of the storm defined by Y^0_3. This field is symmetrical with respect to the plane $z = 0$, and is more localized in radial directions than along the z axis. The absolute maximums of u_N oscillations are reached at the z axis in the points $z' = \pm 1$.

3.2. Tornadoes

Let us now consider a family of fields W^s_j (equation (1)) with $\pi/2 < \theta_2 < \pi$ ($2\pi < \Omega < 4\pi$). Similar to storms and whirls, these fields are highly localized. However, the normal and the radial components of time average energy flux vector S are not vanishing. As a result, lines of energy flux become spiral, provided that $s \neq 0$. That is why we refer to these unique localized fields as elastic tornadoes. They bear some similarities to the fields treated in section 2.1, but their lines of energy flux more closely resemble spirals. As θ_2 tends to π, the step of these spirals decreases. Figure 5 shows a typical energy flux line of such field.

For the fields with $s = 0$ and $\pi/2 < \theta_2 < \pi$ $(2\pi < \Omega < 4\pi)$, the lines of energy flux lie in meridional planes. These fields are intermediate in properties between the elastic storms and the beams with $s = 0$ and $\Omega = 2\pi$ (see section 2.1).

4. Conclusion

Exact time-harmonic solutions of wave equations in an isotropic linear elastic medium are obtained using expansions in plane waves propagating in a given solid angle Ω. They describe superpositions of longitudinal elastic eigenwaves, defined by the spherical harmonics Y_j^s.

Two basically different types of orthonormal elastic beams are presented. The first forms two separate sets of orthonormal beams defined by the spherical harmonics Y_j^s with even and odd j, respectively. These beams are composed from eigenwaves propagating in the solid angle $\Omega = 2\pi$. The second comprises a complete system of orthonormal beams defined by the whole set of spherical harmonics Y_j^s, and for these beams $\Omega \leqslant 2\pi$.

Three different types of localized elastic fields (called storms, whirls, and tornadoes) are also presented. All these fields have a very small and clearly defined core region with maximum intensity of field oscillations. Outside the core, the intensity of oscillations rapidly decreases in all directions. For elastic storms ($\Omega = 4\pi$ and $s = 0$), time average energy flux is identically zero at all points. Whirls ($\Omega = 4\pi$ and $s \neq 0$) and tornadoes ($2\pi < \Omega \leqslant 4\pi$ and $s \neq 0$) have circular and spiral energy flux lines, respectively.

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